

Conformal field theory and the solution of the (quantum) elliptic Calogero-Sutherland system

Edwin Langmann

ABSTRACT. We review the construction of a conformal field theory model which describes anyons on a circle and at finite temperature, including previously unpublished results. This anyon model is closely related to the quantum elliptic Calogero-Sutherland (eCS) system. We describe this relation and how it has led to an explicit construction of the eigenvalues and eigenfunctions of the eCS system.

1. Introduction

The *Sutherland system* is defined by the quantum many body Hamiltonian

$$(1.1) \quad H_N = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \gamma \sum_{1 \leq j < k \leq N} V(x_j - x_k)$$

with the two body interaction potential

$$(1.2) \quad V(r) = \frac{1}{4 \sin^2 \frac{1}{2} r}.$$

It describes an arbitrary number, N , of identical particles moving on a circle of length 2π , where $x_j \in [-\pi, \pi]$ are the particle positions, and $\gamma > -1/4$ is the coupling constant. This model is known to be integrable in the sense that there exist hermitian differential operators $H_N^{(n)}$ of the form

$$(1.3) \quad H_N^{(n)} = \sum_{j=1}^N i^n \frac{\partial^n}{\partial x_j^n} + \text{lower order terms}$$

for $n = 1, 2, \dots$, which include the Hamiltonian, $H_N^{(2)} = H_N$, and which mutually commute, $[H_N^{(n)}, H_N^{(n')}] = 0$ for all $n, n' = 1, 2, \dots, N$. The explicit eigenvalues and eigenfunctions of this model are known since a long time [Su72]. These eigenfunctions are essentially equal to the Jack polynomials which are symmetric functions playing an important role in various areas of mathematics; see [McD79, St89].

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As discovered by Calogero [C71], a more general integrable system can be obtained by replacing the interaction potential above by

$$(1.4) \quad V(r) = \sum_{m \in \mathbb{Z}} \frac{1}{4 \sin^2 \frac{1}{2}(r + i\beta m)}, \quad \beta > 0,$$

which is essentially the Weierstrass elliptic \wp -function with periods 2π and β ;¹ see also [OP77]. The Sutherland system is recovered from this *elliptic Calogero-Sutherland (eCS) system* in the trigonometric limit $\beta \rightarrow \infty$. Much less is known about the eigenfunctions and eigenvalues of the eCS system; see however [DI93, EK94, EFK95, FV95a, FV95b, Sk95, T00, FNP03] for various recent interesting results in this direction.

The topic of this paper is a relation of the eCS system to a particular *conformal field theory (CFT)* model describing *anyons on a circle*,² and the use of this relation to solve the eCS system. These anyons are quantum fields $\phi(x)$ parameterized by a coordinate $x \in [-\pi, \pi]$, and they are characterized by the following relations,

$$(1.5) \quad \phi(x)\phi(y) = e^{\pm i\pi\lambda} \phi(y)\phi(x) \quad \text{for } x \leq y,$$

where $\lambda > 0$ is the so-called *statistics parameter*. Mathematically our anyons can be thought of as generators of a star algebra represented on some Hilbert space, but they are somewhat delicate objects: the product of an anyon and its adjoint is singular as follows,

$$(1.6) \quad \phi(x)\phi(y)^* = \text{const.} (x - y)^{-\lambda} [1 + O(|x - y|)] \quad \text{as } x \rightarrow y.$$

This shows that the anyons are not operators but rather operator valued distributions.

Since anyons commute and anticommute for even and odd integers λ , respectively, they generalize bosons and fermions to anyon statistics.³ It is important to note that, for odd integers $\lambda > 1$, the anyons are fermion-like but nevertheless different from fermions: only for $\lambda = 1$ is Eq. (1.6) consistent with the relations $\phi(x)\phi(y)^* + \phi(y)^*\phi(x) = \delta(x - y)$ characterizing conventional fermions, and for $\lambda > 1$ the distribution $\delta(x - y)$ is replaced by something more singular. Anyons with odd integers $\lambda > 1$ can be regarded as *composite fermions* [J89], and they have been used in effective theories of the edge excitations in a fractional quantum Hall system at filling factor $1/\lambda$; see e.g. [W90, PS01].

One motivation of our work [CL99] was an interesting formal relation between the *fractional quantum Hall effect (FQHE)* and the Sutherland model which we now recall. As is well-known, the Laughlin wave function [Lgh83]

$$(1.7) \quad \tilde{\Psi}_0 = e^{-\sum_{j=1}^N |z_j|^2/4} \prod_{1 \leq j < k \leq N} (z_j - z_k)^{2m+1}$$

is an excellent approximation to the exact ground state of a FQHE system at filling $1/(2m + 1)$, $m = 1, 2, 3, \dots$; $z_j = X_j + iY_j$ with (X_j, Y_j) the electron positions in natural units, and the system is a disc with radius R , $|z_j| \leq R$. On the other hand,

¹ To be precise: $V(z) = \wp(z) + c_0$ with $c_0 = 1/12 - (1/2) \sum_{m=1}^{\infty} \sinh^{-2}[(\beta m)/2]$ [WW62].

² In the physics literature the name ‘anyons’ is usually used for certain 2+1 dimensional quantum fields, and we thus stress that our anyons are in 1+1 dimensions.

³ The name “anyon” refers to *any* phase $e^{\pm i\pi\lambda}$ which appears in the anyon exchange relations.

the ground state of the Sutherland system is [Su72]

$$(1.8) \quad \Psi_0 = e^{ip \sum_{j=1}^N x_j} \prod_{1 \leq j < k \leq N} [\sin \frac{1}{2}(x_j - x_k)]^\lambda$$

with p the total momentum of the center-of-mass motion (p is usually set to zero). A simple computation shows that, if one sets $z_j = R \exp(ix_j)$ in the Laughlin wave function $\tilde{\Psi}_0$, one obtains the Sutherland groundstate Ψ_0 with $\lambda = 2m + 1$ (up to an irrelevant constant and for some value of p). This suggests to interpret Ψ_0 as wave function describing the edge degrees of freedom of a FQHE system. As mentioned above, anyons have been used to construct an effective quantum field theory of the edge excitations of the FQHE, and our original motivation to study anyons and its relation to the Sutherland system was to get a better understanding of the FQHE. This has been a useful guide for our work, even though we were eventually led in quite a different direction. As mentioned in Section 4.3, we still hope that our results will be eventually also useful in the context of FQHE physics.

A central result in our work is the following intriguing fact [CL99] (for earlier work see [BPS94, MP94, H94, AMOS95, I95, MS96, AJL97] and references therein): there exists a self-adjoint operator \mathcal{H} on the anyon Hilbert space so that the commutator of this operator with a product of N anyons is essentially equal to the Sutherland Hamiltonian acting on this product, i.e.

$$(1.9) \quad [\mathcal{H}, \phi(x_1) \cdots \phi_N(x_N)]\Omega = H_N \phi(x_1) \cdots \phi_N(x_N)\Omega$$

where Ω is the vacuum state in the anyon Hilbert space, and the coupling of the Sutherland Hamiltonian H_N in Eq. (1.1) is determined by the statistic parameter of the anyons as follows,

$$(1.10) \quad \gamma = 2\lambda(\lambda - 1).$$

Since one quantum field operator \mathcal{H} accounts for arbitrary particle numbers N of the Sutherland system, we refer to \mathcal{H} as *second quantization of the Sutherland system*. We will also discuss how to use this relation to derive explicit formulas for the eigenvalues and eigenfunctions of the Sutherland system which are equivalent to Sutherland's solution [L01]. Different from Sutherland's method, our approach yields fully explicit formulas for the eigenfunctions, and it can be generalized also to the elliptic case [L00, L04b, L04d].

In the next Section we describe a rigorous construction of anyons and the second quantization of the Sutherland system based on CFT methods due to Graeme Segal [Se81] (see also [PS86]; in Remark 2.2 we will also indicate a possible alternative method of proof using vertex algebras [K98]). We then explain the second quantization of the Sutherland systems using these anyons and how to use it to solve the latter. In Section 3 we explain our generalization of this to the elliptic case [L00, L04b] which has led to an explicit solution of the eCS model [L04d]. We end with remarks on alternative proofs, possible extensions, and open questions in Section 4. In particular, in Section 4.4 we present previously unpublished formulas for operators which presumably provide a second quantization for the higher differential operators $H_N^{(n)}$ of the Sutherland system (for arbitrary n).

This paper is a concise but self-contained review, including results which have not been published before. Many of the proofs are given, and only for a few technically and/or computationally demanding arguments we refer to our original papers.

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2. Anyons at zero temperature: trigonometric case

2.1. Construction of anyons. The starting point of our construction of anyons is the *Heisenberg algebra* which is the algebra with star operation $*$ generated by elements R and $\hat{\rho}(n)$, $n \in \mathbb{Z}$, obeying the following relations

$$(2.1) \quad [\hat{\rho}(m), \hat{\rho}(n)] = m\delta_{m,-n}, \quad [R, \hat{\rho}(n)] = \delta_{n,0}R,$$

and

$$(2.2) \quad \hat{\rho}(n)^* = \hat{\rho}(-n), \quad R^* = R^{-1}$$

for all $m, n \in \mathbb{Z}$, where

$$(2.3) \quad Q \equiv \hat{\rho}(0)$$

has the physical interpretation of a *charge operator*. This is a prominent algebra defining a CFT of bosons where R is an intertwiner between different charge sectors. Mathematically it defines a central extension of the loop group of smooth maps from the circle to $U(1)$; see e.g. [PS86].

The standard highest weight representation of this algebra is fully characterized by a *highest weight state* Ω satisfying

$$(2.4) \quad \hat{\rho}(n)\Omega = 0 \quad \forall n \geq 0.$$

By standard arguments one then constructs a Hilbert space \mathcal{F} containing Ω with an inner product $\langle \cdot, \cdot \rangle$ so that the $*$ is equal to the Hilbert space adjoint; see e.g. [PS86]. We will refer to Ω as *vacuum* and to \mathcal{F} as *anyon Hilbert space*.

Regularized anyons can then be defined as follows,

$$(2.5) \quad \phi_\varepsilon(x) = e^{\sqrt{\lambda} \sum_{n=1}^{\infty} \frac{1}{n} \hat{\rho}(-n) \bar{z}^n} e^{-i\lambda Qx/2} R e^{-i\lambda Qx/2} e^{-\sqrt{\lambda} \sum_{n=1}^{\infty} \frac{1}{n} \hat{\rho}(n) z^n},$$

where

$$(2.6) \quad z \equiv e^{ix-\varepsilon}, \quad \bar{z} \equiv e^{-ix-\varepsilon},$$

and $\varepsilon > 0$ is a regularization parameter [CL99]. To see that this is well-defined, we note that $\phi_\varepsilon(x)$ is proportional to the following unitary operator,

$$(2.7) \quad V_\varepsilon(x) = R \exp\left(-i\lambda Qx - \sqrt{\lambda} \sum_{n \neq 0} \frac{1}{n} \hat{\rho}(n) e^{inx-|n|\varepsilon}\right).$$

Indeed, defining normal ordering $\times \cdots \times$ as usual (see e.g. [CL99]) we have

$$(2.8) \quad \phi_\varepsilon(x) = \times V_\varepsilon(x) \times,$$

and by a straightforward computation using the Hausdorff formula

$$(2.9) \quad e^A e^B = e^{c/2} e^{A+B} = e^c e^B e^A \quad \text{if } [A, B] = c \text{ with } c \in \mathbb{C},$$

one finds that $\phi_\varepsilon(x)$ is proportional to $V_\varepsilon(x)$ with a proportionality constant diverging as $\varepsilon^{-\lambda/2}$ as $\varepsilon \downarrow 0$. The anyons can then be defined as

$$(2.10) \quad \phi(x) \equiv \lim_{\varepsilon \downarrow 0} \phi_\varepsilon(x)$$

where the normal ordering ensures that this limit exists. To check that these anyons obey the correct exchange relations in Eq. (1.5) we again use the Hausdorff relation to compute

$$\phi_\varepsilon(x)\phi_{\varepsilon'}(y) = e^{(\cdots)}\phi_{\varepsilon'}(y)\phi_\varepsilon(x),$$

and we obtain

$$(\cdots) = -\lambda \left(i[x-y] + \sum_{n \neq 0} \frac{1}{n} e^{in(x-y) - |n|(\varepsilon + \varepsilon')} \right) \rightarrow -\lambda i\pi \operatorname{sign}(x-y)$$

as $\varepsilon, \varepsilon' \rightarrow 0$. To show that the anyons are well-defined one can compute the *anyon correlation functions*, i.e., the vacuum expectation values of products of anyons, and convince oneself that they remain well-defined as all regularizations are removed. These correlation functions are also of main physical interest.

PROPOSITION 2.1. *The regularized anyons are bounded Hilbert space operators, and they have well-defined limits $\varepsilon \downarrow 0$ so that all anyon correlations are well-defined. Denoting products of anyons as*

$$(2.11) \quad \Phi_N(\mathbf{x}) \equiv \phi(x_1)\phi(x_2)\cdots\phi(x_N),$$

the non-zero anyon correlation functions are as follows,

$$(2.12) \quad \langle \Omega, \Phi_N(\mathbf{x})^* R^{N-M} \Phi_M(\mathbf{y}) \Omega \rangle = e^{i\frac{1}{2}\lambda(N-M)(X+Y)} F_{N,M}(\mathbf{x}; \mathbf{y})$$

for $N, M = 0, 1, 2, \dots$, where

$$(2.13) \quad F_{N,M}(\mathbf{x}; \mathbf{y}) \equiv \frac{\prod_{1 \leq j < k \leq N} \theta(x_j - x_k)^\lambda \prod_{1 \leq j < k \leq M} \theta(y_k - y_j)^\lambda}{\prod_{j,k=1}^N \theta(x_j - y_k)^\lambda}$$

with

$$(2.14) \quad \theta(r) \equiv \sin \frac{1}{2}r,$$

and

$$(2.15) \quad X \equiv \sum_{j=1}^N x_j, \quad Y \equiv \sum_{j=1}^M y_j.$$

(The proof is by straightforward computations using the defining relations of the Heisenberg algebra and the Hausdorff formula; see [L04b, L04c] for details.)

It is worth noting that the factor R^{N-M} is inserted on the r.h.s. in Eq. (2.12) to get a nonzero result (since $\langle \Omega, A\Omega \rangle = 0$ unless A commutes with Q). In particular,

$$(2.16) \quad \langle \Omega, \phi(x)^* \phi(y) \Omega \rangle = \theta(x-y)^{-\lambda},$$

consistent with Eq. (1.6).

REMARK 2.2. Our regularization has the advantage of producing well-defined operators so that multiplication of anyons becomes unambiguous, but the price we pay is that our regularized anyons are not analytic. An alternative regularization is to analytically continue the anyons to the complex region outside the unit circle and define

$$(2.17) \quad \phi(x) = {}^\times R \exp \left(-i\lambda Qx - \sqrt{\lambda} \sum_{n \neq 0} \frac{1}{n} \hat{\rho}(n) e^{inx} \right) {}^\times, \quad \Im(x) = \varepsilon > 0.$$

These regularized anyons are not operators but only sesquilinear forms, however, working with them makes computations somewhat simpler. For example, Eq. (2.12) remains true for these regularized anyons as it stands (for appropriate complex x_j, y_j), which is not the case for our regularization. Moreover, there exists a mathematical formulation of CFT which allows to give a rigorous meaning to these sesquilinear forms as generators of a so-called vertex algebra; see e.g. [K98].

For simplicity we will mostly ignore this technical issue of regularization in the following discussion.

2.2. Second quantization of the Sutherland model. One natural choice for an anyon Hamiltonian is

$$(2.18) \quad H_0 = \frac{\lambda}{2} Q^2 + \sum_{n=1}^{\infty} \hat{\rho}(-n) \hat{\rho}(n),$$

and it satisfies

$$(2.19) \quad [H_0, \phi(x)] = i \frac{\partial \phi(x)}{\partial x}.$$

We note in passing that this model is a CFT since H_0 is equal to the zero mode L_0 of a $c = 1$ representation of a Virasoro algebra existing in this model (see e.g. [K98, PS86]): the generators L_n are the Fourier modes of $\times \rho(x)^2 \times / 2$ with

$$(2.20) \quad \rho(x) = \sqrt{\lambda} Q + \sum_{n \neq 0} \hat{\rho}(n) e^{inx}$$

the boson field in position space. Eq. (2.19) suggests to interpret H_0 as second quantization of (minus the) the momentum operator $i\partial/\partial x$. It is interesting to try to find another anyon Hamiltonian which corresponds to a second quantization of the second derivative $-\partial^2/\partial x^2$. By straightforward computations one finds that the operator

$$(2.21) \quad \mathcal{H} = \sqrt{\lambda} W^3 - \frac{3\lambda - 2}{12} Q + (1 - \lambda) \mathcal{C}$$

with

$$(2.22) \quad W^3 = \frac{1}{3} \int_{-\pi}^{\pi} \frac{dx}{2\pi} \times \rho(x)^3 \times, \quad \mathcal{C} = \sum_{n=1}^{\infty} n \hat{\rho}(-n) \hat{\rho}(n)$$

obeys the relations

$$(2.23) \quad [\mathcal{H}, \phi(x)] = -\frac{\partial^2}{\partial x^2} \phi(x) + \dots$$

where the dots stand for terms which annihilate the vacuum, $(\dots)\Omega = 0$. This seems to be the closest one can get to a second quantization of $-\partial^2/\partial x^2$. While W^3 above is a local operator, it is interesting to note that \mathcal{C} is non-local: in position space it can be written in the following suggestive form

$$(2.24) \quad \mathcal{C} = -\lim_{\varepsilon \downarrow 0} \int \frac{dx}{2\pi} \int \frac{dy}{2\pi} \frac{\rho(x)\rho(y)}{4 \sin^2 \frac{1}{2}(x-y+i\varepsilon)}.$$

It is interesting to compute the commutator of \mathcal{H} with a product of anyon operators. By a straightforward but tedious computation one finds the relations in Eq. (2.27). We summarize these results as follows.

PROPOSITION 2.3. *There exist mutually commuting self-adjoint operators Q, H_0 and \mathcal{H} on the anyon Hilbert space with the following relations,*

$$(2.25) \quad [Q, \Phi_N(\mathbf{x})] = N\Phi_N(\mathbf{x}),$$

$$(2.26) \quad [H_0, \Phi_N(\mathbf{x})] = \sum_{j=1}^N i \frac{\partial}{\partial x_j} \Phi_N(\mathbf{x}),$$

$$(2.27) \quad [\mathcal{H}, \Phi_N(\mathbf{x})]\Omega = \left(-\sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{j < k} 2\lambda(\lambda - 1)V(x_j - x_k) \right) \Phi_N(\mathbf{x})\Omega,$$

with $\Phi_N(\mathbf{x})$ in Eq. (2.11) and $V(r)$ in Eq. (1.2).

(More details and full proofs can be found in [CL99].)

The relation of main interest for us is in Eq. (2.27), but we formulated this proposition so as to suggest a more general result: as discussed in more detail Section 4.4, we believe that there exists a second quantization of all the commuting differential operators $H_N^{(n)}$ of the Sutherland system mentioned in the introduction also for $n > 2$, and we also have a specific conjecture for these operators.

2.3. Solution of the Sutherland system. Proposition 2.3 provides a nice method to construct eigenfunctions of the Sutherland system.

COROLLARY 2.4. *Let η be an eigenstate of the CFT operator \mathcal{H} with eigenvalue \mathcal{E} , $\mathcal{H}\eta = \mathcal{E}\eta$, and with charge N , $Q\eta = N\eta$. Then the inner product of η with the state $\Phi_N(\mathbf{x})\Omega$,*

$$(2.28) \quad \Psi_\eta(\mathbf{x}) \equiv \langle \Omega, \Phi(\mathbf{x})^* \eta \rangle,$$

is an eigenfunction of the Sutherland Hamiltonian H_N in Eqs. (1.1), (1.2), (1.10) with the same eigenvalue \mathcal{E} ,

$$(2.29) \quad H_N \Psi_\eta(\mathbf{x}) = \mathcal{E} \Psi_\eta(\mathbf{x}).$$

PROOF. We compute $\langle \Omega, \Phi_\eta(\mathbf{x})^* \mathcal{H} \eta \rangle = \langle \mathcal{H} \Phi_\eta(\mathbf{x}) \Omega, \eta \rangle$ in two different ways: firstly, using Eq. (2.27) and the highest weight condition

$$(2.30) \quad \mathcal{H}\Omega = 0$$

yields the l.h.s. of Eq. (2.29), and secondly using $\mathcal{H}\eta = \mathcal{E}\eta$ gives the r.h.s. \square

Remarkably, Eq. (2.27) can also be used to construct enough eigenstates η of \mathcal{H} to recover all the eigenfunctions of H_N . The idea is to take the Fourier transform of Eq. (2.27). For that one has to remember an important detail: the anyons are not periodic in general but obey

$$(2.31) \quad \phi(x + 2\pi) = e^{-i\lambda Q\pi} \phi(x) e^{-i\lambda Q\pi}$$

with Q the charge operator obeying $[Q, \phi(x)] = \phi(x)$ and $Q\Omega = 0$. Thus we need to remove the non-periodic factor from the anyons before we can take the Fourier transform,

$$(2.32) \quad \hat{\phi}(n) = \int_{-\pi}^{\pi} \frac{dx}{2\pi} e^{i\lambda Qx/2} \phi(x) e^{i\lambda Qx/2} e^{inx}, \quad n \in \mathbb{Z}.$$

A straightforward computation then yields the following result.

PROPOSITION 2.5. *The following product of Fourier transformed anyons,*

$$(2.33) \quad \hat{\Phi}_N(\mathbf{n}) \equiv \hat{\phi}(n_1)\hat{\phi}(n_2)\cdots\hat{\phi}(n_N), \quad \mathbf{n} = (n_1, \dots, n_N) \in \mathbb{Z}^N,$$

obeys the relation

$$(2.34) \quad [\mathcal{H}, \hat{\Phi}_N(\mathbf{n})]\Omega = \mathcal{E}_0(\mathbf{n})\hat{\Phi}_N(\mathbf{n})\Omega - \gamma \sum_{1 \leq j < k \leq N} \sum_{\nu=1}^{\infty} \nu \hat{\Phi}_N(\mathbf{n} + \nu \mathbf{E}_{jk})\Omega$$

with

$$(2.35) \quad \mathcal{E}_0(\mathbf{n}) = \sum_{j=1}^N \tilde{n}_j^2, \quad \tilde{n}_j = n_j + \frac{1}{2}\lambda(2N + 1 - 2j),$$

and

$$(2.36) \quad (\mathbf{E}_{jk})_\ell \equiv \delta_{j\ell} - \delta_{k\ell}, \quad j, k, \ell = 1, 2, \dots, N.$$

REMARK 2.6. As will be explained below, the $\mathcal{E}_0(\mathbf{n})$ in Eq. (2.35) are equal to the eigenvalues of the Sutherland Hamiltonian. As is well-known, they are remarkably similar to the eigenvalues of the non-interacting case: the Sutherland interaction only changes the momenta n_j to the so-called *pseudo-momenta* \tilde{n}_j . It is important to note the following arbitrariness in the definition of the pseudo-momenta: if $\Psi(\mathbf{x})$ is an eigenfunction of the Sutherland Hamiltonian H_N with eigenvalue $\mathcal{E}_0(\mathbf{n}) = \sum_{j=1}^N \tilde{n}_j^2$, then

$$(2.37) \quad \tilde{\Psi}(\mathbf{x}) = e^{-ip \sum_{j=1}^N x_j} \Psi(\mathbf{x})$$

is also an eigenfunction with eigenvalue

$$(2.38) \quad \mathcal{E}'_0(\mathbf{n}) = \sum_{j=1}^N (\tilde{n}'_j)^2, \quad \tilde{n}'_j = \tilde{n}_j - p.$$

Thus the pseudo-momenta can be shifted by changing the center-of-mass motion which is, of course, a trivial change. Our definition of pseudo-momenta here differs from the one in [Su72, L00, L04b] by the constant $p = N\lambda/2$. As we will see, this later choice puts the center-of-mass to rest; see Eq. (3.20) below.

PROOF. We observe that

$$(2.39) \quad \hat{\Phi}_N(\mathbf{n})\Omega = \int_{[-\pi, \pi]^N} \frac{d^N x}{(2\pi)^N} \Phi_N(\mathbf{x}) e^{i\tilde{\mathbf{n}} \cdot \mathbf{x}} \Omega$$

with $\tilde{\mathbf{n}} = (\tilde{n}_1, \dots, \tilde{n}_N)$, where the shifts in the Fourier modes \tilde{n}_j come from the factors $\exp(i\lambda Q x_j/2)$ introduced to compensate the non-periodicity of the anyons. Eq. (2.34) then follows if we recall Eq. (1.1), with the first term on the r.h.s. coming from the derivative terms in H_N and partial integrations, and the second from the interaction using

$$(2.40) \quad \frac{1}{4 \sin^2 \frac{1}{2}(x_j - x_k + i\varepsilon)} = - \sum_{\nu=1}^{\infty} \nu e^{i\nu(x_j - x_k + i\varepsilon)}$$

with $\varepsilon \downarrow 0$ our regularization parameter. \square

The proof shows that the regularization of the anyons is not just a technicality but crucial to get correct results.

A simple computation then yields the solution of the Sutherland system in terms of anyon correlation functions.

THEOREM 2.7. *The CFT operator \mathcal{H} has eigenstates $\eta_N(\mathbf{n})$ with corresponding eigenvalues $\mathcal{E}_0(\mathbf{n})$ given in Eq. (2.35) provided that*

$$(2.41) \quad n_1 \geq n_2 \geq \dots \geq n_N.$$

These eigenstates are

$$(2.42) \quad \eta_N(\mathbf{n}) = \sum_{\mathbf{m}} \alpha_{\mathbf{n}}(\mathbf{m}) \hat{\Phi}_N(\mathbf{m}) \Omega \quad \text{with} \quad \alpha_{\mathbf{n}}(\mathbf{n}) = 1$$

where the coefficients are given by

$$(2.43) \quad \begin{aligned} \alpha_{\mathbf{n}}(\mathbf{m}) = & \delta(\mathbf{m}, \mathbf{n}) + \sum_{s=1}^{\infty} \gamma^s \sum_{j_1 < k_1} \sum_{\nu_1=1}^{\infty} \nu_1 \cdots \\ & \times \cdots \sum_{j_s < k_s} \sum_{\nu_s=1}^{\infty} \nu_s \frac{\delta(\mathbf{m}, \mathbf{n} + \sum_{r=1}^s \nu_r \mathbf{E}_{j_r k_r})}{\prod_{\ell=1}^s [\mathcal{E}_0(\mathbf{n} + \sum_{r=1}^{\ell} \nu_r \mathbf{E}_{j_r k_r}) - \mathcal{E}_0(\mathbf{n})]}. \end{aligned}$$

Thus the eigenfunctions of the Sutherland model are

$$(2.44) \quad \Psi(\mathbf{x}; \mathbf{n}) = \langle \Omega, \Phi_N(\mathbf{x})^* \tilde{\eta}_N(\mathbf{n}) \rangle$$

with \mathbf{n} satisfying Eq. (2.41), and the corresponding eigenvalues are $\mathcal{E}_0(\mathbf{n})$ in Eq. (2.35).

PROOF. It is easy to see that the ansatz in Eq. (2.42) implies

$$(2.45) \quad \mathcal{H} \eta_N(\mathbf{n}) = \mathcal{E}_{\mathbf{n}} \eta_N(\mathbf{n})$$

if $\alpha_{\mathbf{n}}(\mathbf{m})$ obeys

$$(2.46) \quad [\mathcal{E}_0(\mathbf{m}) - \mathcal{E}_{\mathbf{n}}] \alpha_{\mathbf{n}}(\mathbf{m}) = \gamma \sum_{j < k} \sum_{\nu=1}^{\infty} \nu \alpha_{\mathbf{n}}(\mathbf{m} - \nu \mathbf{E}_{jk}).$$

Demanding $\alpha_{\mathbf{n}}(\mathbf{n}) = 1$ we get $\mathcal{E}_{\mathbf{n}} = \mathcal{E}_0(\mathbf{n})$ (this is true since the previous relation system of equations has triangular structure in some natural sense). We now make the ansatz

$$\alpha_{\mathbf{n}}(\mathbf{m}) = \sum_{s=0}^{\infty} \gamma^s \alpha_{\mathbf{n}}^{(s)}(\mathbf{m}), \quad \alpha_{\mathbf{n}}^{(0)}(\mathbf{m}) = \delta(\mathbf{m}, \mathbf{n}),$$

and from the relation above we get the recursion relations

$$\alpha_{\mathbf{n}}^{(s)}(\mathbf{m}) = \frac{1}{[\mathcal{E}_0(\mathbf{m}) - \mathcal{E}_0(\mathbf{n})]} \sum_{j < k} \sum_{\nu=1}^{\infty} \nu \alpha_{\mathbf{n}}^{(s-1)}(\mathbf{m} - \nu \mathbf{E}_{jk}).$$

It is straightforward to solve this by induction and thus obtain the result in Eq. (2.43).

To see that all the eigenstates are well-defined we note that

$$m_j = (\mathbf{n} + \underline{\mu})_j = n_j - \sum_{k < j} \mu_{kj} + \sum_{k > j} \mu_{jk}$$

where we use the notation

$$(2.47) \quad \mathbf{m} - \mathbf{n} = \underline{\mu} = \sum_{j < k} \mu_{jk} \mathbf{E}_{jk},$$

where it is important to note that only \mathbf{m} 's with non-negative integers μ_{jk} occur above. With that one can prove a highest weight conditions showing that there are

always only a finite number of states $\hat{\Phi}_N(\mathbf{n} + \underline{\mu})\Omega$ different from zero (see Appendix C.3 in [CL99]). It is also important to note that

$$(2.48) \quad \mathcal{E}_0(\mathbf{n} + \underline{\mu}) - \mathcal{E}_0(\mathbf{n}) = \sum_{j=1}^N \left(2 \sum_{k=j+1}^N \mu_{jk} [n_j - n_k + (k-j)\lambda] + \left[\sum_{k < j} \mu_{kj} - \sum_{k > j} \mu_{jk} \right]^2 \right),$$

which is a sum of positive terms if the condition in (2.41) holds true. This shows that there are no zero denominators. Moreover, the sum in Eq. (2.43) is always finite due to the Kronecker delta. This shows that all $\eta_N(\mathbf{n})$ are finite linear combination of states $\hat{\Phi}_N(\mathbf{m})\Omega$ and thus well-defined. \square

It is straightforward to compute the functions

$$(2.49) \quad \hat{F}_N(\mathbf{x}; \mathbf{n}) \equiv \langle \Omega, \Phi_N(\mathbf{x})^* \hat{\Phi}_N(\mathbf{n}) \Omega \rangle$$

explicitly by taking the Fourier transform of the function $F_{NN}(\mathbf{x}; \mathbf{y})$ in Eq. (2.13) w.r.t. \mathbf{y} [L01]; see Eq. (3.16) below. In Ref. [L01] we also showed that Theorem 2.7 reproduces Sutherland's solution [Su72].

Note that Theorem 2.7 implies that the CFT operators

$$(2.50) \quad \tilde{\Phi}_N(\mathbf{n}) = \sum_{\mathbf{m}} \alpha_{\mathbf{n}}(\mathbf{m}) \hat{\Phi}_N(\mathbf{m})$$

are eigenstates of \mathcal{H} in the following sense,

$$(2.51) \quad [\mathcal{H}, \tilde{\Phi}_N(\mathbf{n})] \Omega = \mathcal{E}_0(\mathbf{n}) \tilde{\Phi}_N(\mathbf{n}) \Omega.$$

We thus can construct many more explicit formulas for eigenfunctions of the Sutherland Hamiltonian H_N as follows,

$$(2.52) \quad \langle \Omega, R^{N-M} \Phi_N(\mathbf{x})^* \tilde{\Phi}_M(\mathbf{n})^* \Omega \rangle,$$

where N and M can be different and the charge changing factor R^{N-M} is inserted to get a non-zero result. Thus there seems to exist many different explicit formulas for each eigenfunction of the Sutherland system. We plan to give a more detailed discussion of these formulas elsewhere.

3. Anyons at finite temperature: elliptic case

We now describe a generalization of the results in the previous section to the elliptic case. We start with the heuristic argument which led us to these results.

As is well-known, the fermion two-point correlation function at zero temperature is $1/\theta(x-y)$ with $\theta(r) = \sin(r/2)$ (Eq. (2.16) for $\lambda = 1$), and at finite temperature this trigonometric function is replaced by

$$(3.1) \quad \theta(r) = \sin(\tfrac{1}{2}r) \prod_{n=1}^{\infty} (1 - 2q^{2n} \cos r + q^{4n}), \quad q = e^{-\beta/2}$$

which is essentially the Jacobi Theta function ϑ_1 ,⁴ with $\beta > 0$ the inverse temperature; see e.g. [CH87] and references therein. As discussed in Section 2.1, at zero temperature the function $\theta(r) = \sin(r/2)$ is the building block of all anyon

⁴ To be precise: $\theta(r) = \vartheta_1(r/2)/[2q^{1/4} \prod_{n=1}^{\infty} (1 - q^{2n})]$ [WW62].

correlation functions, and in the derivation of Eq. (2.27) the interaction potential $V(r)$ arises from $\theta(r)$ as follows [CL99],

$$(3.2) \quad V(r) = -\frac{\partial^2}{\partial r^2} \log \theta(r).$$

It is not difficult to see that, if we insert $\theta(r)$ in (3.1), $V(r)$ in Eq. (3.2) becomes identical with the elliptic function in Eq. (1.4). This suggested to us that one should be able to generalize the results in the previous section to the elliptic case by using anyons at finite temperature (which seem to be equivalent to vertex operators on a torus; see e.g. [B88]).

3.1. Finite temperature anyons and the eCS system. To construct anyons at finite temperature one could use the representation of anyons in Section 2.2, with the vacuum expectation value replaced by the usual Gibbs state for the anyon Hamiltonian in Eq. (2.18),

$$(3.3) \quad \langle A \rangle_\beta \equiv \frac{1}{\mathcal{Z}} \text{Tr}_0(e^{-\beta H_0} A)$$

for any operator A in the Heisenberg algebra,⁵ with the usual normalization constant $\mathcal{Z} = \text{Tr}_0 \exp(-\beta H_0)$. The trace Tr_0 here is only over the charge zero sector in the anyon Hilbert space (this turns out to be the correct choice).

It is convenient to use an alternative construction based on a reducible representation of the Heisenberg algebra obtained by the well-known trick of doubling the degrees of freedom: we start with a standard highest weight representation of two commuting copies of the Heisenberg algebra,

$$(3.4) \quad [\hat{\rho}_A(n), \hat{\rho}_B(m)] = m\delta_{m,-n}\delta_{A,B}, \quad [\hat{\rho}_A(n), R_B] = \delta_{n,0}\delta_{A,B}R_A$$

and

$$(3.5) \quad R_A^* = R^{-1}, \quad \hat{\rho}_A(n)^* = \hat{\rho}_A(-n),$$

for $A, B = 1, 2$ and $m, n \in \mathbb{Z}$, and with a highest weight vector Ω such that

$$(3.6) \quad \hat{\rho}_A(n)\Omega = 0 \quad \forall n \geq 0, \quad A = 1, 2.$$

Then

$$\pi(R) \equiv R_1, \quad \pi(\hat{\rho}(0)) \equiv \hat{\rho}_1(0)$$

$$(3.7) \quad \pi(\hat{\rho}(n)) \equiv c_n \hat{\rho}_1(n) + s_n \hat{\rho}_2(-n) \quad \forall n \neq 0$$

obviously defines a representation π of the Heisenberg algebra, provided that the parameters c_n and s_n obey the relations

$$(3.8) \quad |c_n|^2 - |s_n|^2 = 1, \quad \overline{c_n} = c_{-n}, \quad \overline{s_n} = s_{-n} \quad \forall n \neq 0$$

with the bar denoting complex conjugation.

A particular choice for c_n and s_n allows us to represent the Gibbs state as vacuum expectation value in this reducible representation.

PROPOSITION 3.1. *For any element A in the Heisenberg algebra, the vacuum expectation value in the representation π above with*

$$(3.9) \quad |c_n|^2 = \frac{1}{1 - q^{2n}}, \quad |s_n|^2 = \frac{q^{2n}}{1 - q^{2n}}, \quad q = e^{-\beta/2}$$

⁵ Note that A can be essentially any operator on the zero temperature anyon Hilbert space.

is identical with the finite temperature expectation value of A ,

$$(3.10) \quad \langle \Omega, \pi(A) \Omega \rangle = \langle A \rangle_\beta,$$

with the r.h.s. defined in Eq. (3.3).

To simplify notation we write in the following A short for $\pi(A)$.

We then define finite temperature *anyons* as in Eq. (2.17), with the definition of normal ordering changed in an obvious manner (see Eq. (25) in [L04b]).⁶

The results in Sections 2.2 and 2.3 generalize in a surprisingly straightforward manner:

PROPOSITION 3.2. *Propositions 2.1 and 2.3 hold true as they stands also at finite temperature $\beta < \infty$, provided that the functions $\theta(r)$ and $V(r)$ in Eqs. (2.14) and (1.2) are replaced by their elliptic generalizations in Eq. (3.1) and (1.4).*

(For more details and proofs see [L04b].)

However, the construction in Section 2.4 cannot be generalized immediately due to a seemingly innocent detail: the second quantized eCS Hamiltonian \mathcal{H} does *not* obey the highest weight condition in Eq. (2.30). To realize that this condition is crucial it is helpful to note that Proposition 3.2 is not restricted to the finite temperature representations but holds true for *any* irreducible representation π , i.e., c_n and s_n obeying (3.8) can be (essentially) arbitrary, and all results remain true if we take

$$(3.11) \quad \theta(r) = \exp\left(-\frac{ir}{2} - \sum_{n=1}^{\infty} \frac{1}{n} [|c_n|^2 e^{inr} + |s_n|^2 e^{-inr}]\right)$$

and

$$(3.12) \quad V(r) = - \sum_{n=1}^{\infty} n (|c_n|^2 e^{inr} + |s_n|^2 e^{-inr}).$$

This suggests that the integrable elliptic case must have a special property. This property is as follows.

LEMMA 3.3. *For the irreducible representation π defined above and all elements A in the Heisenberg algebra, the following conditions holds true:*

$$(3.13) \quad \langle \Omega, [\mathcal{H}, A] \Omega \rangle = 0$$

if and only if c_n and s_n are as in Eq. (3.9) for some $\beta > 0$.

(The proof can be found in [L04b], Appendix B.)

It is interesting to note that the non-local part \mathcal{C} in the explicit formula (2.21) of the operator \mathcal{H} is representation dependent and cannot be written in terms of the Heisenberg algebra alone; see Eq. (58) in [L04b]. Thus while it seems at first that the method of doubling degrees of freedom to construct finite temperature representations is only a convenient trick, we feel that there is more to it.

As we now show, Eq. (3.13) is enough to construct the eigenfunctions of the eCS system explicitly.

⁶ It is not necessary to change the normal ordering prescription since it only amounts to a change by finite constants, but it is convenient since this leads to somewhat simpler formulas.

3.2. Solution of the eCS system. We insert $A = \Phi_N(\mathbf{x})^* \hat{\Phi}_N(\mathbf{n})$ into Eq. (3.13) and obtain

$$\langle [\mathcal{H}, \Phi_N(\mathbf{x})] \Omega, \hat{\Phi}_N(\mathbf{n}) \Omega \rangle = \langle \Phi_N(\mathbf{x}) \Omega, [\mathcal{H}, \hat{\Phi}_N(\mathbf{n})] \Omega \rangle$$

where we used the Leibniz rule for the commutator and that \mathcal{H} is self-adjoint.

We thus see that, while Corollary 2.4 is no longer true, we can bypass it. All we need is the following generalization of Proposition 2.5.

PROPOSITION 3.4. *Proposition 2.5 holds true also at finite temperature but with Eq. (2.34) replaced by*

$$(3.14) \quad [\mathcal{H}, \hat{\Phi}_N(\mathbf{n})] \Omega = \mathcal{E}_0(\mathbf{n}) \hat{\Phi}_N(\mathbf{n}) \Omega - \gamma \sum_{\nu \in \mathbb{Z}} S_\nu \hat{\Phi}(\mathbf{n} + \nu \mathbf{E}_{jk}) \Omega$$

where

$$(3.15) \quad S_0 = 0, \quad S_\nu = \frac{\nu}{1 - q^{2\nu}} \quad \text{and} \quad S_{-\nu} = \frac{\nu q^{2\nu}}{1 - q^{2\nu}} \quad \forall \nu > 0$$

and $\mathcal{E}_0(\mathbf{n})$ is as in Eq. (2.35).

(The proof is as in the zero temperature case, only now we use Eq. (3.12) and (3.9) for taking into account the interaction term [L04b].)

With that it is possible to generalize the arguments leading to the solution of the Sutherland system in Theorem 2.7. This provides an alternative route to the explicit solution of the eCS system announced in [L04d].

THEOREM 3.5. *Let*

$$(3.16) \quad \hat{F}(\mathbf{x}; \mathbf{n}) \equiv \mathcal{P}(\mathbf{x}; \mathbf{n}) \Psi_0(\mathbf{x}),$$

with

$$(3.17) \quad \mathcal{P}(\mathbf{x}; \mathbf{n}) \equiv \prod_{j=1}^N \left[\oint_{\mathcal{C}_j} \frac{d\xi_j}{2\pi\xi_j} \xi_j^{n_j} \right] \frac{\prod_{1 \leq j < k \leq N} \Theta(\xi_j/\xi_k)^\lambda}{\prod_{j,k=1}^N \Theta(e^{ix_j}/\xi_k)^\lambda}$$

where

$$(3.18) \quad \Theta(\xi) \equiv (1 - \xi) \prod_{m=1}^{\infty} [(1 - q^{2m}\xi)(1 - q^{2m}/\xi)]$$

and the integration contours are nested circles in the complex plane enclosing the unit circle,

$$(3.19) \quad \mathcal{C}_j : \quad \xi_j = e^{\varepsilon j} e^{iy_j}, \quad -\pi \leq y_j \leq \pi, \quad 0 < \varepsilon < \beta/N,$$

and

$$(3.20) \quad \Psi_0(\mathbf{x}) \equiv e^{iN\lambda \sum_{j=1}^N x_j/2} \prod_{1 \leq j < k \leq N} \theta(x_j - x_k)^\lambda$$

with the function $\theta(r)$ in Eq. (3.1). Then for each $\mathbf{n} \in \mathbb{Z}^N$ satisfying the condition in (2.41), there is an eigenfunction $\Psi(\mathbf{x}; \mathbf{n})$ of the eCS Hamiltonian in Eqs. (1.1), (1.4) given by

$$(3.21) \quad \Psi(\mathbf{x}; \mathbf{n}) = \sum_{\mathbf{m}} \alpha_{\mathbf{n}}(\mathbf{m}) \hat{F}(\mathbf{x}; \mathbf{m})$$

with the coefficients

$$(3.22) \quad \alpha_{\mathbf{n}}(\mathbf{m}) = \delta(\mathbf{m}, \mathbf{n}) + \sum_{s=1}^{\infty} \gamma^s \sum_{j_1 < k_1} \sum_{\nu_1 \in \mathbb{Z}} S_{\nu_1} \cdots \\ \times \cdots \sum_{j_s < k_s} \sum_{\nu_s \in \mathbb{Z}} S_{\nu_s} \frac{\delta(\mathbf{m}, \mathbf{n} + \sum_{r=1}^s \nu_r \mathbf{E}_{j_r k_r})}{\prod_{r=1}^s [\mathcal{E}_0(\mathbf{n} + \sum_{\ell=1}^r \nu_{\ell} \mathbf{E}_{j_{\ell} k_{\ell}}) - \mathcal{E}_{\mathbf{n}}]_{\mathbf{n}}}$$

with $\mathcal{E}_0(\mathbf{n})$ in Eq. (2.35), S_{ν} in Eq. (3.15), and $\mathcal{E}_{\mathbf{n}}$ the corresponding eigenvalue determined by the following implicit equation:

$$(3.23) \quad \mathcal{E}_{\mathbf{n}} = \mathcal{E}_0(\mathbf{n}) - \sum_{s=2}^{\infty} \gamma^s \sum_{j_1 < k_1} \sum_{\nu_1 \in \mathbb{Z}} S_{\nu_1} \cdots \\ \times \cdots \sum_{j_s < k_s} \sum_{\nu_s \in \mathbb{Z}} S_{\nu_s} \frac{\delta(\mathbf{0}, \sum_{r=1}^s \nu_r \mathbf{E}_{j_r k_r})}{\prod_{r=1}^s [\mathcal{E}_0(\mathbf{n} + \sum_{\ell=1}^r \nu_{\ell} \mathbf{E}_{j_{\ell} k_{\ell}}) - \mathcal{E}_{\mathbf{n}}]_{\mathbf{n}}}.$$

The double square brackets above are to indicate that all terms which are undefined for $\mathcal{E}_{\mathbf{n}} = \mathcal{E}_0(\mathbf{n})$, i.e. where $\sum_{\ell=1}^r \nu_{\ell} \mathbf{E}_{j_{\ell} k_{\ell}} = \mathbf{0}$, should be discarded,

$$(3.24) \quad \frac{1}{[\mathcal{E}_0(\mathbf{m}) - \mathcal{E}_{\mathbf{n}}]_{\mathbf{n}}} \equiv [1 - \delta(\mathbf{n}, \mathbf{m})] \frac{1}{[\mathcal{E}_0(\mathbf{m}) - \mathcal{E}_{\mathbf{n}}]}.$$

For simplicity we ignore the problem of so-called *resonances* here; see [L04a]. The generalization of our argument above to prove this result is not completely obvious. An outline of a somewhat empirical derivation of this result appeared in [L04d], and we plan to give a self-contained proof in a future revision of Ref. [L04a].

To get eigenfunctions in one-to-one correspondence with the Sutherland case one should impose the restriction in Eq. (2.41), even though this restriction does not seem necessary in the derivation of this result. However, we suspect that it is necessary to ensure that Eq. (3.23) has a well-defined solution.

From Eq. (3.23) one can obtain by straightforward computations a fully explicit formula for the eigenvalues as follows,

$$(3.25) \quad \mathcal{E}_{\mathbf{n}} = \mathcal{E}_0(\mathbf{n}) + \sum_{n=1}^{\infty} (-1)^n \sum_{k_0, \dots, k_{n-1}=0}^{\infty} \delta(n-1, \sum_{j=1}^{n-1} j k_j) \\ \times \delta(n, \sum_{j=1}^{n-1} k_j) \binom{n-1}{k_0, \dots, k_{n-1}} \prod_{j=0}^{n-1} [G_j(\mathbf{n})]^{k_j}$$

with

$$(3.26) \quad G_k(\mathbf{n}) \equiv \sum_{s=2}^{\infty} \gamma^s \sum_{j_1 < k_1} \sum_{\nu_1 \in \mathbb{Z}} S_{\nu_1} \cdots \sum_{j_s < k_s} \sum_{\nu_s \in \mathbb{Z}} S_{\nu_s} \sum_{\ell_1, \dots, \ell_s=0}^{\infty} \delta(k, \sum_{r=1}^s \ell_r) \\ \times \frac{\delta(\sum_{r=1}^s \nu_r \mathbf{E}_{j_r k_r}, \mathbf{0})}{\prod_{r=1}^{s-1} [\mathcal{E}_0(\mathbf{n} + \sum_{\ell=1}^r \nu_{\ell} \mathbf{E}_{j_{\ell} k_{\ell}}) - \mathcal{E}_0(\mathbf{n})]_{\mathbf{n}}^{1+\ell_s}},$$

and there are similarly explicit formulas for the coefficients $\alpha_{\mathbf{n}}(\mathbf{m})$ (we plan to give the latter in a revised version of Ref. [L04a]). All results stated above hold true at least in the sense of formal power series in q^2 , but there exist results which seem to imply that all our series have a finite radius of convergence in q^2 [KT02].

Note that the functions $\mathcal{P}(\mathbf{x}; \mathbf{n})$ are symmetric in the variables $z_j = e^{ix_j}$, and they can be expanded as Laurent series. For $q = 0$ they become symmetric polynomials which are, in fact, equal to the so-called Jack polynomials [McD79, St89]; see [L01]. It is also interesting to note that $\Psi_0(\mathbf{x})$ for $q = 0$ is the well-known ground state wave function of the Sutherland model [Su72], but it is *not* an eigenfunction of the eCS Hamiltonian for $q > 0$. We thus see that $q > 0$ makes the solution much more involved, even though it is still possible to write it down explicitly.

4. Final remarks

1. While we found the solution method for the eCS as described in this paper, the final result can be proven without using CFT: Theorem 3.5 can also be derived from the functional identity

$$(4.1) \quad H_N(\mathbf{x})F_{N,N}(\mathbf{x}; \mathbf{y}) = H_N(\mathbf{y})F_{N,N}(\mathbf{x}; \mathbf{y})$$

for the anyon correlation function defined in Eqs. (2.13) and (3.1) and the eCS Hamiltonians in Eqs. (1.1) and (1.4) but acting on different arguments \mathbf{x} and \mathbf{y} , as indicated. While Eq. (4.1) is a simple consequence of Proposition 3.2 and Lemma 3.3, it can also be proven by brute-force computations using a well-known functional relation of the Weierstrass elliptic functions [L04a]. While this method of proof is elementary it seems ad-hoc, and it would have been difficult for us to get the details right without knowing the CFT results discussed in this paper. In [L04c] we were able to find many more such identities using these CFT results.

2. The solution method based on a remarkable identity as in (4.1) works for many more quantum integrable systems than the ones discussed here.⁷

3. The motivation for starting this project was the FQHE, and even though it seems now that our results are mainly of interest in the context of integrable systems, we still hope that they will be eventually relevant for the FQHE. We now shortly describe some ideas in this direction.

As indicated in the introduction, we believe that the second quantized Sutherland Hamiltonian \mathcal{H} for odd integers λ describe the FQHE system at filling $1/\lambda$: starting with a realistic many-body Hamiltonian for the FQHE describing fermions in two dimensions in a strong magnetic field and interacting via the Coulomb potential, one can map this to a 1D interaction fermions system by projection into the lowest Landau level; we believe that \mathcal{H} describes a fixed point of the renormalization group applied to this latter Hamiltonian. If one could substantiate this, the exact eigenstates of \mathcal{H} computed in Section 2.3 would become relevant in the theory of the FQHE. It would be also interesting to find corresponding Hamiltonians describing more complicated filling fractions and corresponding to more complicated ground states of composite fermions [J89] — perhaps these would also be interesting integrable systems.

4. As discussed after Proposition 2.3, it is natural to conjecture that there exist self-adjoint operators \mathcal{H}_n on the anyon Hilbert space so that

$$(4.2) \quad [\mathcal{H}_n, \phi(x)]\Omega = i^n \frac{\partial^n}{\partial x^n} \phi(x)\Omega$$

⁷ M. Hallnäs and E. Langmann, in preparation.

for all non-negative integers n . In 1999 we⁸ constructed an operator valued generating functional for these operators by generalizing the method in [CL99] to all orders. This functional is defined as follows,

$$(4.3) \quad \mathcal{W}(a) = \sum_{n=0}^{\infty} \frac{(-ia)^n}{n!} \mathcal{H}_n$$

and thus obeys

$$(4.4) \quad [\mathcal{W}(a), \phi(x)]\Omega = \phi(x+a)\Omega$$

as a formal power series in a . We found

$$(4.5) \quad \mathcal{W}(a) = \sum_{s=0}^{\infty} \frac{iw_s(a)}{2[\cos(\frac{1}{2}a)]^\lambda \tan(\frac{1}{2}a)^\lambda} \tilde{\mathcal{W}}_s(a)$$

with

$$(4.6) \quad \tilde{\mathcal{W}}_s(a) \equiv \int_{-\pi}^{\pi} \frac{dx}{2\pi} \left[\mathcal{V}_-(x; a) \frac{\partial^s}{\partial x^s} \mathcal{V}_+(x; a) - I\delta_{s,0} \right],$$

where

$$(4.7) \quad \mathcal{V}_\pm(x; a) \equiv \exp\left(-i\sqrt{\lambda} \sum_{k=0}^{\infty} \frac{a^{k+1}}{(k+1)!} \rho_\pm^{(k)}(x)\right)$$

and

$$(4.8) \quad \rho_\pm(x) \equiv \frac{1}{2}\sqrt{\lambda}Q + \sum_{n \neq 0} \hat{\rho}(\mp n) e^{\mp i n x}$$

are the creation and annihilation parts of the boson fields, $\rho_\pm^{(k)}(x) = \partial^k \rho_\pm(x) / \partial x^k$, and the coefficients $w_s(a)$ are determined by the recursion relations

$$(4.9) \quad w_0(a) \equiv 1, \quad w_s(a) \equiv - \sum_{k=0}^{s-1} v_{s-k}(a) w_k(a)$$

where

$$(4.10) \quad v_k(a) \equiv \sum_{\ell=k}^{\infty} \frac{1}{\lambda} \binom{\lambda}{\ell+1} [-\tan(\frac{1}{2}a)]^\ell \frac{1}{k!\ell!} \frac{d^\ell}{dc^\ell} \left[\frac{2 \arctan(c)}{(1+c^2)} \right] \Big|_{c=0}.$$

Note that the operator $\tilde{\mathcal{W}}_0(a)$ is equal to $\int_{-\pi}^{\pi} dx \stackrel{\times}{\times} \phi(x+a) \phi(x)^* \stackrel{\times}{\times}$ and gives rise to the local part of the \mathcal{H}_n , and the $\tilde{\mathcal{W}}_{s>0}(a)$ give non-local corrections. It is straightforward to write a MATLAB program which computes all \mathcal{H}_n explicitly (this is possible since $w_s(a) = O(a^s)$, and thus only the operators $\tilde{\mathcal{W}}_s(a)$ for $s < n$ contribute to \mathcal{H}_n .) One thus can check that $\mathcal{H}_0 = Q$, $\mathcal{H}_1 = H_0$, $\mathcal{H}_2 = \mathcal{H}$ as given in Section 2.2, and the next operator is

$$(4.11) \quad \begin{aligned} \mathcal{H}_3 = \int_{-\pi}^{\pi} \frac{dx}{2\pi} & \left(\stackrel{\times}{\times} \left[\frac{\lambda}{4} \rho(x)^4 + \frac{1}{4} (\rho'(x))^2 - \frac{1}{8} (3\lambda - 2) \rho(x)^2 \right] \stackrel{\times}{\times} \right. \\ & + \frac{3i}{2} \sqrt{\lambda} (\lambda - 1) [\rho_-(x) (\rho_+(x)^2)' + \rho_-(x)^2 \rho_+(x)'] \\ & \left. - \frac{1}{2} (2\lambda - 1) (\lambda - 1) \rho_-(x) \rho_+(x)'' \right) \end{aligned}$$

where the terms in the first line are local and the others are non-local corrections.

⁸ E. Langmann, unpublished.

It is easy to show that all \mathcal{H}_n are self-adjoint and annihilate the vacuum Ω . It is natural to expect that these operators \mathcal{H}_n provide a second quantization of the higher conservation laws $H_N^{(n)}$ of the Sutherland system, but we have not been able to prove this result.

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MATHEMATICAL PHYSICS, KTH PHYSICS, ALBANOVA, SE-106 91 STOCKHOLM, SWEDEN
E-mail address: langmann@theophys.kth.se